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Commutator representations of differential calculi on the quantum group $SU_q(2)$

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Abstract

Let (Γ, d) be the $3D$-calculus or the $4D_\pm$-calculus on the quantum group $SU_q(2)$. We describe all pairs (π, F) of a *-representation π of $\mathcal{O}(SU_q(2))$ and of a symmetric operator F on the representation space satisfying a technical condition concerning its domain such that there exist a homomorphism of first order differential calculi which maps $\mathrm{d}x$ into the commutator $[\mathrm{i}F, \pi(x)]$ for $x \in \mathcal{O}(SU_q(2))$. As an application commutator representations of the two-dimensional left-covariant calculus on Podles quantum 2-sphere S^2_{qc} with $c = 0$ are given. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

There are various ways to develop a noncommutative differential calculus on a given *-algebra $\mathcal{A}$. Some approaches are based on derivations [3,7] of the algebra $\mathcal{A}$, while others consider differential forms as the basic objects. In Alain Connes' program [2] of noncommutative geometry, the fundamental concept for the quantized calculus is the K-cycle. Other names for closely related notions are Fredholm modules and spectral triples. If we omit technical subtleties, then the underlying idea of all these concepts is easy to explain: One has a *-representation π of the algebra $\mathcal{A}$ and a self-adjoint operator F on

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the representation space, and the differentiation of an algebra element a is defined by the commutator of the operators $\mathrm{i}F$ and $\pi(a)$, where i is the imaginary unit.

On the other hand, for quantum groups there is a well-developed theory of covariant differential calculi which was initiated by Woronowicz [15], see also the monograph [6]. The relations between this theory and Connes' approach via K-cycles or spectral triples are still open. Let us be more specific and consider the simplest nontrivial compact quantum group $SU_q(2)$. Then we have three distinguished left-covariant differential calculi on the Hopf $*$-algebra $\mathcal{O}(SU_q(2))$: the $3D$-calculus and the $4D_\pm$-calculi. All three calculi are due to Woronowicz [14,15]. The $4D_\pm$-calculi are even bicovariant, but we shall not use this here. These calculi have a number of nice properties, and there is a common belief that they are favoured candidates for the study of noncommutative geometry on the quantum group $SU_q(2)$. Thus it seems to be natural to ask whether or not one of these calculi can be described by means of a K-cycle or more generally by the commutators $[\mathrm{i}F, \pi(\cdot)]$ for some $*$-representation π of $\mathcal{O}(SU_q(2))$ and some symmetric operator F.

Under an additional technical condition called admissibility we described all such pairs (π, F) which represent the $3D$-calculus or the $4D_\pm$-calculus on $SU_q(2)$. The main results about this matter are stated as Theorem 1–3 in Section 4, while the proofs of these results are postponed to Section 7. It turns out that the $3D$-calculus can be faithfully represented as a commutator. Some properties of such commutator representations of the $3D$-calculus and some examples are treated in Section 5. An application to the Podles quantum sphere S^2_{qc} with $c = 0$ is sketched in Section 6. The $4D_\pm$-calculus does not admit a faithful admissible commutator representation, because any such representation passes to a three-dimensional left-covariant quotient calculus. Further, we prove that neither for the $3D$-calculus nor for the $4D_\pm$-calculus there exists a nontrivial commutator representation (π, F) such that all operators $[\mathrm{i}F, \pi(x)]$, $x \in \mathcal{A}$, are bounded. This shows in particular that none of these calculi can be given by means of a K-cycle in the sense of Connes.

In Section 2 we collect some basic definitions on differential calculi and some simple facts needed later. In Section 3 we repeat the structure of a general $*$-representation of the $*$-algebra $\mathcal{O}(SU_q(2))$ and some facts about the $3D$- and the $4D_\pm$-calculi on $SU_q(2)$. Further, we develop the two-dimensional calculus on the Podles sphere S^2_{pc} for $c = 0$ as the induced FODC of the $3D$-calculus on $SU_q(2)$.

Let us fix some general notation. We use the Sweedler notation $\Delta(x) = x_{(1)} \otimes x_{(2)}$ for the comultiplication of a Hopf algebra element x. Let $x(n)y$ denote the equation which is obtained by multiplying equation (n) by x from the left and by y from the right. Throughout we set

$$\lambda := q - q^{-1} \quad \text{and} \quad \lambda_+ := q + q^{-1}.$$

2. Commutator representations of first order differential calculi

Let $\mathcal{A}$ be a complex $*$-algebra with unit element. The involution of $\mathcal{A}$ is denoted by $*$. A *first order differential calculus* (abbreviated as FODC) over $\mathcal{A}$ is a pair (Γ, d)

of an $\mathcal{A}$-bimodule Γ with a linear mapping $\mathrm{d} : \mathcal{A} \to \Gamma$ such that the following two conditions hold:

(i) d satisfies the Leibniz rule $\mathrm{d}(xy) = x \cdot \mathrm{d}y + \mathrm{d}x \cdot y$ for $x, y \in \mathcal{A}$,

(ii) $\Gamma = \mathrm{Lin}\{x \cdot \mathrm{d}y \cdot z; x, y, z \in \mathcal{A}\}$.

A *first order differential* *-*calculus* (briefly, a *-FODC) over $\mathcal{A}$ is a FODC (Γ, d) equipped with an involution $* : \Gamma \to \Gamma$ of the complex vector space Γ such that

(iii) $(x \cdot \mathrm{d}y \cdot z)^* = z^* \cdot \mathrm{d}(y^*) \cdot x^*$ for $x, y, z \in \mathcal{A}$.

By a *homomorphism* of a FODC (Γ_1, d_1) into a FODC (Γ_2, d_2) over $\mathcal{A}$ we mean a linear mapping $\rho : \Gamma_1 \to \Gamma_2$ such that $\rho(x \cdot \mathrm{d}_1 y \cdot z) = x \cdot d_2 y \cdot z$ for $x, y, z \in \mathcal{A}$. A *homomorphism of a* *-FODC $(\Gamma_1, \mathrm{d}_1, *_1)$ into a *-FODC $(\Gamma_1, \mathrm{d}_2, *_2)$ is a FODC homomorphism $\rho : \Gamma_1 \to \Gamma_2$ such that $\rho(\omega^{*_1}) = \rho(w)^{*_2}$ for $\omega \in \Gamma_1$. If no ambiguity can arise, we denote a FODC (Γ, d) or a *-FODC $(\Gamma, \mathrm{d}, *)$ by Γ.

Now let $\mathcal{A}$ be a Hopf *-algebra. A *-FODC Γ over $\mathcal{A}$ is called *left-covariant* if there exists a linear mapping $\varphi : \Gamma \to \mathcal{A} \otimes \Gamma$ such that $\varphi(x \cdot \mathrm{d}y) = \Delta(x)(\mathrm{id} \otimes \mathrm{d})\Delta(y)$ for all $x, y \in \mathcal{A}$. Suppose Γ is a left-covariant *-FODC. We define

$$\omega_\Gamma(x) := S(x_{(1)})\,\mathrm{d}x_{(2)}, \quad x \in \mathcal{A}, \quad \text{and} \quad \mathcal{R}_\Gamma := \{x \in \ker \epsilon : \omega_\Gamma(x) = 0\}.$$

It is well known [6,15] that $\mathcal{R}_\Gamma$ is a right ideal of $\mathcal{A}$ which characterizes the left-covariant FODC Γ up to isomorphism.

Suppose that π is a *-representation of the *-algebra $\mathcal{A}$ by bounded operators on a Hilbert space $\mathcal{H}$ and F is a symmetric linear operator on $\mathcal{H}$ with dense domain $\mathcal{D}(F)$. Let us assume that there exists a linear subspace $\mathcal{D}_F$ of $\mathcal{D}(F)$ such that $\mathcal{D}_F$ is dense in $\mathcal{H}$ and $\pi(\mathcal{A})\mathcal{D}_F \subseteq \mathcal{D}(F)$. Let $\Gamma_{\pi,F}$ denote the linear span of operators

$$\pi(x)(F\pi(y) - \pi(y)F)\pi(z)\lceil\mathcal{D}_F, \quad x, y, z \in \mathcal{A},$$

where the symbol $\lceil\mathcal{D}_F$ denotes the restriction of the corresponding operator to $\mathcal{D}_F$. It is clear that $\Gamma_{\pi,F}$ is an $\mathcal{A}$-bimodule with left and right action of an element $x \in \mathcal{A}$ given by multiplication by $\pi(x)$ from the left and the right, respectively. Define a linear mapping $\mathrm{d}_{\pi,F} : \mathcal{A} \to \Gamma_{\pi,F}$ and an antilinear mapping $* : \Gamma_{\pi,F} \to \Gamma_{\pi,F}$ by

$$\mathrm{d}_{\pi,F}(x) := (\mathrm{i}F\pi(x) - \pi(x)\mathrm{i}F)\lceil\mathcal{D}_F, \quad x \in \mathcal{A},$$

where i is the imaginary unit, and

$$T^* := \sum_j \pi(z_j^*)(\pi(y_j^*)F - F\pi(y_j^*))\pi(x_j^*)\lceil\mathcal{D}_F,$$

where

$$T = \sum_j \pi(x_j)(F\pi(y_j) - \pi(y_j)F)\pi(z_j)\lceil\mathcal{D}_F \in \Gamma_{\pi,F}.$$

Using the facts that $\mathcal{D}_F$ is dense in $\mathcal{H}$, the operator F is symmetric and π is a *-representation it is easy to check that the mapping $T \to T^*$ is well-defined (that is, $T^* = 0$ when $T = 0$). Further, it is not difficult to verify that the triple $(\Gamma_{\pi,F}, \mathrm{d}_{\pi,F}, *)$ is a *-FODC over $\mathcal{A}$. We call it the **-FODC associated with the pair* (π, F) and denote it simply by $\Gamma_{\pi,F}$. (With a few modifications concerning the domains the preceding construction carries over to unbounded *-representations π as well. We shall not need this in this paper, because we consider the coordinate *-algebra $\mathcal{O}(SU_q(2))$ which has only bounded *-representations.) The above notion is closely related to Alain Connes' concept of a K-cycle. If in addition F is a self-adjoint operator with compact resolvent and the commutator $[\pi(x), F]$ is bounded for any $x \in \mathcal{A}$, the pair (π, F) is called a *K-cycle* over the *-algebra $\mathcal{A}$.

Now let Γ be an arbitrary *-FODC over $\mathcal{A}$ and let π and F be as above. We shall say that the pair (π, F) is a *commutator representation* of the *-FODC Γ if there exists a homomorphism ρ of the *-FODC Γ to the *-FODC $\Gamma_{\pi,F}$ associated with (π, F). If ρ is injective, then (π, F) is called a *faithful* commutator representation of Γ. A slight reformulation of this definition is given by the following lemma.

Lemma 1. *A pair (π, F) as above is a commutator representation of the *-FODC Γ if and only if the relation $\Sigma_j x_j \, \mathrm{d}y_j = 0$ in Γ with $x_j, y_j \in \mathcal{A}$ always implies that $\sum_j \pi(x_j)(\mathrm{i}F\pi(y_j) - \pi(y_j)\mathrm{i}F)\lceil\mathcal{D}_F = 0$.*

Proof. The only if part is trivial. If this condition is fulfilled, then there exists a well-defined linear map $\rho : \Gamma \to \Gamma_{\pi,F}$ such that $\rho(\sum_j x_j \, \mathrm{d}y_j) = \sum_j \pi(x_j)(\mathrm{i}F\pi(y_j) - \pi(y_j)\mathrm{i}F)\lceil\mathcal{D}_F$. One easily checks that ρ is a homomorphism of Γ to $\Gamma_{\pi,F}$. □

Let (π, F) be a pair of a *-representation π of $\mathcal{A}$ and a symmetric linear operator as above. For $x \in \mathcal{A}$ we define the linear operator

$$\Omega_{\pi,F}(x) := (\pi(S(x_{(1)}))F\pi(x_{(2)}) - \varepsilon(x)F)\lceil\mathcal{D}_F. \tag{1}$$

The following very simple observations are needed in the proofs of the main theorems later.

Lemma 2. *Suppose that (π, F) is a commutator representation of the left-covariant FODC Γ and let ρ denote a FODC homomorphism of Γ to $\Gamma_{\pi,F}$. Then we have $\rho(\omega_\Gamma(x)) = \mathrm{i}\Omega_{\pi,F}(x)$ for all $x \in \mathcal{A}$. In particular, if x belongs to the right ideal R_Γ associated with Γ, then $\Omega_{\pi,F}(x) = 0$.*

Proof. Using the definitions of $\omega_\Gamma(x)$ and $\Omega_{\pi,F}(x)$ and the fact that ρ is a FODC homomorphism we compute

$$\begin{aligned}\rho(\omega_\Gamma(x)) &= \rho(S(x_{(1)}))\rho(\mathrm{d}x_{(2)}) = \rho(S(x_{(1)}))\,\mathrm{d}_{\pi,F}(x_{(2)}) \\ &= \pi(S(x_{(1)}))(\mathrm{i}F\pi(x_{(2)}) - \pi(x_{(2)})\mathrm{i}F)\lceil\mathcal{D}_F = \mathrm{i}\Omega_{\pi,F}(x). \qquad \square\end{aligned}$$

Lemma 3. *Let $\rho : \Gamma_1 \to \Gamma_2$ be a homomorphism of the left-covariant FODC Γ_1 into another FODC Γ_2. Suppose that $\mathcal{B}$ is a subset of $\ker\epsilon$ such that $\rho(\omega_{\Gamma_1}(b)) = 0$ for all $b \in \mathcal{B}$. Let Γ_0 denote the quotient FODC of Γ_1 whose associated right ideal $\mathcal{R}_{\Gamma_0}$ is generated by $\mathcal{R}_{\Gamma_1}$ and $\mathcal{B}$. Then ρ passes to a homomorphism of the quotient FODC Γ_0 to the FODC Γ_2.*

Proof. If Γ_f denotes the universal FODC over $\mathcal{A}$, then any other left-covariant FODC Γ over $\mathcal{A}$ is isomorphic to the quotient FODC $\Gamma_f/\mathcal{A}\omega_{\Gamma_f}(\mathcal{R}_\Gamma)$ (see, for instance, [6, Proposition 14.1]). This implies that the FODC Γ_0 is isomorphic to the quotient FODC $\Gamma_1/\mathcal{A}\omega_{\Gamma_1}(\mathcal{R}_{\Gamma_0})$. Therefore, it is sufficient to prove that $\rho(x\omega_{\Gamma_1}(by)) = 0$ for all $b \in \mathcal{B}$ and $x, y \in \mathcal{A}$. Indeed, using the facts that ρ is a bimodule homomorphism and $\omega_{\Gamma_1}(by) = S(y_{(1)})\omega_{\Gamma_1}(b)y_{(2)}$ (see formula (14.3) in [6]) we obtain

$$\rho(x\omega_{\Gamma_1}(by)) = \rho(xS(y_{(1)}))\rho(\omega_{\Gamma_1}(b))\rho(y_{(2)}) = 0. \qquad \square$$

3. Preliminaries on the quantum group $SU_q(2)$

From now let $\mathcal{A}$ be the coordinate $*$-algebra $\mathcal{O}(SU_q(2))$ of the compact quantum group $SU_q(2)$ [6,8,13,14]. In what follows we shall assume that $0 < q < 1$. The generators of $\mathcal{A}$ are the four entries a, b, c, d of the fundamental matrix and the involution of $\mathcal{A}$ is given by

$$a^* = d, \quad b^* = -qc, \quad c^* = -q^{-1}b, \quad d^* = a. \tag{2}$$

3.1. Star representations of $\mathcal{O}(SU_q(2))$

Suppose that π is an arbitrary $*$-representation of the $*$-algebra $\mathcal{A} = \mathcal{O}(SU_q(2))$. From Proposition 4.19 in [6] or from the description of the irreducible $*$-representations in [13] it follows that up to unitary equivalence the $*$-representation π is given by the following operator-theoretic model:

Let v and w be unitary operators on Hilbert spaces $\mathcal{G}$ and $\mathcal{H}_0$, respectively. Put $\mathcal{H} = \oplus_{n=0}^{\infty}\mathcal{H}_n$, where $\mathcal{H}_n = \mathcal{H}_0$ for all $n \in \mathbb{N}_0$. For $\eta \in \mathcal{H}_0$, let η_n denote the vector of $\mathcal{H}$ which has η as its nth component and zero otherwise. The $*$-representation π acts on the direct sum Hilbert space $\mathcal{G} \oplus \mathcal{H}$ and it is determined by the formulas

$$\pi(a) = v, \quad \pi(d) = v^*, \quad \pi(b) = \pi(c) = 0 \quad \text{on} \quad \mathcal{G}, \tag{3}$$

$$\begin{aligned} &\pi(a)\eta_n = \lambda_n\eta_{n-1}, \quad \pi(d)\eta_n = \lambda_{n+1}\eta_{n+1}, \quad \pi(c)\eta_n = q^n w\eta_n, \\ &\pi(b)\eta_n = -q\pi(c)^*\eta_n = -q^{n+1}w^*\eta_n \end{aligned} \tag{4}$$

for $\eta \in \mathcal{H}_0$ and $n \in \mathbb{N}_0$, where we have set $\eta_{-1} := 0$ and

$$\lambda_n := (1 - q^{2n})^{1/2}, \quad n \in \mathbb{N}_0.$$

Note that the $*$-representation π is parametrized by the two unitaries v and w.

Let (π, F) be a commutator representation of a $*$-FODC Γ of $\mathcal{A} = \mathcal{O}(SU_q(2))$. Then (π, F) is called *admissible* if there exist linear subspaces $\mathcal{E} \subseteq \ker \pi(c)$ and $\mathcal{D} \subseteq \ker \pi(a)$ such that $\pi(d)\mathcal{E} \subseteq \mathcal{E}$, $\pi(a)\mathcal{E} \subseteq \mathcal{E}$, $\pi(b)\mathcal{D}_0 \subseteq \mathcal{D}_0$, $\pi(d)\mathcal{D}_0 \subseteq \mathcal{D}_0$ and the domain $\mathcal{D}_F := \mathcal{E} + \mathrm{Lin}\{\pi(d^n)\mathcal{D}_0;\, n \in \mathbb{N}_0\}$ is contained in $\mathcal{D}(F)$ and is a core for the (unbounded symmetric) operator F. The dense linear subspace $\mathcal{D}_F$ is then invariant under all operators $\pi(x)$, $x \in \mathcal{A}$, and so D_F may be taken as the domain used in the definition of the commutator representation (π, F) given in Section 2. The admissibility of a representation (π, F) is a technical condition which is essentially used in the proofs of the main results in Section 6. This condition becomes rather natural if it is considered in terms of the above model for the *-representation π. Clearly, we have $\mathcal{G} = \ker \pi(c)$, $\mathcal{H}_0 = \ker \pi(a)$ and $\mathcal{H}_n = \pi(d^n)\mathcal{H}_0$. That (π, F) is admissible means that there are linear subspaces $\mathcal{E}$ and $\mathcal{G}$ and $\mathcal{D}_0$ of $\mathcal{H}_0$ such that $v\mathcal{E} = \mathcal{E}$, $w\mathcal{D}_0 = \mathcal{D}_0$ and $\mathcal{D}_F := \mathcal{E} \oplus \mathcal{D}$ is a core for the operator F, where $\mathcal{D} := \mathrm{Lin}\{\mathcal{D}_n; n \in \mathbb{N}_0\}$ and $\mathcal{D}_n := \{\eta_n; \eta \in \mathcal{D}_0\}$ is the nth shift of the domain $\mathcal{D}_0$.

3.2. The 3D-calculus

The $3D$-calculus was introduced by Woronowicz [14]. A short approach was given in [12]. Apart from Section 6, we use only the following facts concerning the $3D$-calculus.

The right ideal of the $3D$-calculus is generated by the following six elements (see [14, (2.27)] or [6, (14.23)]):

$$b^2, c^2, bc, (a-1)b, (a-1)c, q^2a + d - (q^2+1). \tag{5}$$

The three 1-forms $\omega_0 := \omega_\Gamma(b), \omega_2 := \omega_\Gamma(c), \omega_1 := \omega_\Gamma(a)$ form a basis of the vector space ${}_{\mathrm{inv}}\Gamma$ and the bimodule structure of Γ is determined by the following commutation relations (see [14, p. 135] or [6, p. 499]):

$$q\omega_j a = a\omega_j,\quad \omega_j b = qb\omega_j,\quad q\omega_j c = c\omega_j,\quad \omega_j d = qd\omega_j \quad \text{for } j = 0, 2, \tag{6}$$

$$q^2\omega_1 a = a\omega_1,\quad \omega_1 b = q^2 b\omega_1,\quad q^2\omega_1 c = c\omega_1,\quad \omega_1 d = q^2 d\omega_1. \tag{7}$$

3.3. The $4D_\pm$-calculus

In order to describe the $4D_\pm$-calculus $\Gamma_\pm$, we first restate some facts developed in Section 14.2.4 in [6]. The quantum tangent space $\mathcal{T}_\pm$ of the $4D_\pm$-calculus is expressed therein in terms of the generators E, F, K, K^{-1} of the Hopf algebra $\hat{U}_q(sl_2)$ (see [6, p. 57]). Let $\varepsilon_+ := \varepsilon$ and let ε_- be the character of the algebra $\mathcal{O}(SU_q(2))$ such that $\varepsilon_-(a) = \varepsilon_-(d) = -1$ and $\varepsilon_-(b) = \varepsilon_-(c) = 0$. Then $\mathcal{T}_\pm$ is spanned by the four linear functionals

$$X_1 = \varepsilon_\pm K^{-2} - \varepsilon,\quad X_2 = q^{1/2}\varepsilon_\pm FK^{-1},\quad X_3 = q^{-1/2}\varepsilon_\pm EK^{-1},$$
$$X_4 = \varepsilon_\pm K^2 + \lambda q^{-1}\varepsilon_\pm FE - \varepsilon$$

and we have

$$\Delta(X_1) = \varepsilon \otimes X_1 + X_1 \otimes \varepsilon_\pm K^{-2}, \tag{8}$$

$$\Delta(X_j) = \varepsilon \otimes X_j + X_j \otimes \varepsilon_\pm + X_1 \otimes X_j, \quad j = 2, 3, \tag{9}$$

$$\begin{aligned}\Delta(X_4) = {} & \varepsilon \otimes X_4 + \lambda^2 q^{-1} X_1 \otimes \varepsilon_\pm FE + X_4 \otimes \varepsilon_\pm K^2 \\ & + \lambda^2 q^{-1/2}(X_2 \otimes \varepsilon_\pm EK + X_3 \otimes \varepsilon_\pm KF).\end{aligned} \tag{10}$$

There is a dual pairing of Hopf algebras $\check{U}_q(sl_2)$ and $\mathcal{O}(SU_q(2))$ determined on the generators by the equations

$$\langle K, a\rangle = q^{-1/2},\quad \langle K, d\rangle = q^{1/2},\quad \langle E, c\rangle = \langle F, b\rangle = 1 \text{ and zero otherwise.} \tag{11}$$

Let $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ be a basis of the vector space ${}_{\mathrm{inv}}\Gamma$ such that $(X_i, \omega_j) = \delta_{ij}, i, j = 1, \dots, 4$. We set $\epsilon = +1$ for the $4D_+$-calculus and $\epsilon = -1$ for the $4D_-$- calculus. From (11) we derive

$$\langle X_1, a\rangle = \epsilon q - 1, \quad \langle X_1, d\rangle = \epsilon q^{-1} - 1, \quad \langle X_2, b\rangle = \langle X_3, c\rangle = \epsilon, \tag{12}$$

$$\langle X_4, a\rangle = \epsilon q^{-1} - 1 + \epsilon\lambda^2 q^{-1}, \quad \langle X_4, d\rangle = \epsilon q - 1. \tag{13}$$

The other pairings of X_i with matrix generators a, b, c, d vanish. Since $\langle X, x\rangle = (X, \omega(x))$ for $X \in \mathcal{T}_\pm$ and $X \in \mathcal{A}$ (see [6, formula (14.9)]), we therefore obtain

$$\omega_\Gamma(a) = (\epsilon q - 1)\omega_1 + (\epsilon q^{-1} - 1 + \epsilon\lambda^2 q^{-1})\omega_4, \quad \omega_\Gamma(b) = \epsilon\omega_2, \tag{14}$$

$$\omega_\Gamma(d) = (\epsilon q^{-1} - 1)\omega_1 + (\epsilon q - 1)\omega_4, \quad \omega_\Gamma(c) = \epsilon\omega_3. \tag{15}$$

From the general theory of left-covariant FODC we recall that

$$\omega_i x = \sum_j x_{(1)} f^i_j(x_{(2)})\omega_j, \quad x \in \mathcal{A}, \tag{16}$$

where the functionals f_i^j are determined by the equation $\Delta(X_i) = \varepsilon \otimes X_i + \sum_j X_j \otimes f_i^j$.

From the formulas (8)–(10) we therefore read off that the nonzero functionals f_j^i are

$$f_1^1 = \varepsilon_\pm K^{-2}, \quad f_2^1 = X_2, \quad f_3^1 = X_3, \quad f_4^1 = \lambda^2 q^{-1}\varepsilon_\pm FE, \quad f_2^2 = f_3^3 = \varepsilon_\pm,$$
$$f_4^2 = \lambda^2 q^{-1/2}\varepsilon_\pm EK, \quad f_4^3 = \lambda^2 q^{-1/2}\varepsilon_\pm KF, \quad f_4^4 = \varepsilon_\pm K^2.$$

Inserting these functionals into (14) and (15) and using the dual pairing (11) we obtain the following list of commutation relations which describes the bimodule structure of the $4D_\pm$-calculus:

$$\begin{aligned}
&\omega_1 a = \epsilon q a\omega_1 + \epsilon b\omega_3 + \epsilon\lambda^2 q^{-1} a\omega_4, \quad \omega_1 b = \epsilon q^{-1} b\omega_1 + \epsilon a\omega_2,\\
&\omega_1 c = \epsilon q c\omega_1 + \epsilon d\omega_3 + \epsilon\lambda^2 q^{-1} c\omega_4, \quad \omega_1 d = \epsilon q^{-1} d\omega_1 + \epsilon c\omega_2,\\
&\omega_2 a = \epsilon a\omega_2 + \epsilon\lambda^2 q^{-1} b\omega_4, \quad \omega_2 b = \epsilon b\omega_2,\\
&\omega_2 c = \epsilon c\omega_2 + \epsilon\lambda^2 q^{-1} d\omega_4, \quad \omega_2 d = \epsilon d\omega_2,\\
&\omega_3 a = \epsilon a\omega_3, \quad \omega_3 b = \epsilon b\omega_3 + \epsilon\lambda^2 q^{-1} a\omega_4,\\
&\omega_3 c = \bar{\omega}_3, \quad \omega_3 d = \epsilon d\omega_3 + \epsilon\lambda^2 q^{-1} c\omega_4,\\
&\omega_4 a = \epsilon q^{-1} a\omega_4, \quad \omega_4 b = \epsilon q b\omega_4, \quad \omega_4 c = \epsilon q^{-1} c\omega_4, \quad \omega_4 d = \epsilon q d\omega_4.
\end{aligned}$$

Further, we note that

$$\omega_1^* = -\omega_1, \quad \omega_2^* = -\omega_3, \quad w_4^* = -\omega_4 \tag{17}$$

and that the right ideal $\mathcal{R}_{\Gamma_\pm}$ admits the following nine generators (see [15, p. 132] or [6, p. 504]):

$$\begin{aligned}
&b^2, c^2, b(a-d), c(a-d), a^2 + q^2 d^2 - (1+q^2)(ad + q^{-1}bc),\\
&\quad z_\pm b, z_\pm c, z_\pm(a-d), z_\pm(q^2 a + d - (q^2+1)),
\end{aligned} \tag{18}$$

where

$$z_\pm := q^2 a + d - \epsilon(q^3 + q^{-1}).$$

Let $\mathcal{R}_{\pm,3}$ denote the right ideal of $\mathcal{A}$ generated by $\mathcal{R}_{\Gamma\pm}$ and the single element $a + \epsilon q d$. Since $S(x)^* \in \mathcal{R}_{\pm,3}$ for $x \in \mathcal{R}_{\pm,3}$, there exists a left-covariant *-FODC $\Gamma_{\pm,3}$ of $\mathcal{A}$ which is a quotient of the $4D_\pm$-calculus $\Gamma_\pm$ and has the associated right ideal $\mathcal{R}_{\pm,3}$ (by Proposition 14.6 in [6]). Since $\langle X_1, a + \epsilon q d\rangle = 0$ by (12) and (13), it is not difficult to check that the quantum tangent space of $\Gamma_{\pm,3}$ is spanned by the three functionals X_1, X_2, X_3. Hence the FODC $\Gamma_{\pm,3}$ has dimension 3. If we set $\omega_4 = 0$ in the above formulas for the $4D_\pm$- calculus $\Gamma_\pm$, then we obtain the corresponding formulas for the FODC $\Gamma_{\pm,3}$. In particular, we see that the *-FODC $\Gamma_{+,3}$ gives the classical first order differential calculus on $SU(2)$ in the limit $q \to 1$. Thus the *-FODC $\Gamma_{\pm,3}$ on $SU_q(2)$ seems to be of interest in itself.

Remark 1. *As noted in* [4], *the* $4D_\pm$*-calculus* $\Gamma_\pm$ *is an irreducible bicovariant* FODC, *because it is derived from the fundamental corepresentation of* $SL_q(2)$ *which is irreducible. By definition this means that* $\Gamma_\pm$ *has no non-trivial* bicovariant *quotient* FODC. *However, as we have seen,* $\Gamma_{\pm,3}$ *is a nontrivial* left-covariant *quotient* FODC *of* $\Gamma_\pm$.

3.4. The two-dimensional calculus on the quantum sphere S_q^2

The considerations of this section are only needed in Section 6.

Let $\mathcal{O}(S_q^2)$ denote the unital *-subalgebra of $\mathcal{A} = \mathcal{O}(SU_q(2))$ generated by the elements

$$x_+ := ba, \quad x_- := cd, \quad y_0 := bc.$$

In order to shorten some formulas it is occasionally convenient to replace y_0 by the element

$$x_0 := \lambda_+ bc + 1 = \lambda_+ y_0 + 1.$$

From the formulas for the comultiplication of the generators a, b, c, d in $\mathcal{O}(SU_q(2))$ we get

$$\Delta(x_+) = a^2 \otimes x_+ + q^{-1} b^2 \otimes x_- + \lambda_+ ba \otimes y_0 + ba \otimes 1, \tag{19}$$

$$\Delta(x_-) = qc^2 \otimes x_+ + d^2 \otimes x_- + \lambda_+ cd \otimes y_0 + cd \otimes 1, \tag{20}$$

$$\Delta(y_0) = ac \otimes x_+ + db \otimes x_- + x_0 \otimes y_0 + bc \otimes 1. \tag{21}$$

That is, we have $\Delta(\mathcal{O}(S_q^2)) \subseteq \mathcal{A} \otimes \mathcal{O}(S_q^2)$ and hence $\mathcal{O}(S_q^2)$ is a left quantum space of $\mathcal{A}$ (that is, a left $\mathcal{A}$-comodule algebra) with coaction given by the restriction of the comultiplication. It is well known that $\mathcal{O}(S_q^2)$ is the coordinate algebra of Podles' quantum 2-sphere S_{qc}^2 in the case $c = 0$ [9]. (Note that the quantum 2-spheres in [1,9,10] are right quantum spaces, while we consider the corresponding left quantum spaces here.) The generators x_+, x_-, y_0 satisfy the relations

$$x_+ x_- - q^2 x_- x_+ = (q^2 - 1) y_0^2, \quad x_+ x_- - q^4 x_- x_+ = (1 - q^2) q y_0, \tag{22}$$

$$x_+ y_0 = q^2 x_+ y_0, \quad q^2 x_- y_0 = y_0 x_-. \tag{23}$$

In fact, the algebra $\mathcal{O}(S_q^2)$ can be also characterized as the abstract unital algebra with generators x_+, x_-, y_0 and definining relations (22) and (23). Since q is real, the algebra $\mathcal{O}(S_q^2)$ is a *-algebra with involution determined by $(x_+)^* = x_-$ and $(y_0)^* = y_0$.

As shown by Podles [10], the quantum space S_q^2 carries a unique two-dimensional *-FODC. For the application given in Section 6 it is crucial that this calculus is induced from the $3D$-calculus of the quantum group $SU_q(2)$. This fact has been known to the author since several years (in fact, since the writing of Apel and Schmüdgen [1]) and also to others (S. Majid, P. Podles). Since I could not find this result in the literature, we shall derive it in this section. In order to do so we first repeat some more facts on the $3D$-calculus from Section 14.1.3 in [6]. Let (Γ, d) be the $3D$-calculus on $\mathcal{O}(SU_q(2))$ and let Γ_2 denote the induced *-FODC of Γ on the *-subalgebra $\mathcal{O}(S_q^2)$.

The quantum tangent space $\mathcal{T}$ of the $3D$-calculus has the three basis elements

$$X_0 := q^{-1/2}FK, \quad X_2 := q^{1/2}EK, \quad X_1 := (1-q^{-2})^{-1}(\varepsilon - K^4)$$

satisfying

$$\Delta(X_j) = \varepsilon \otimes X_j + X_j \otimes K^2 \text{ for } j = 0, 2 \text{ and } \Delta(X_1) = \varepsilon \otimes X_1 + X_1 \otimes K^4. \tag{24}$$

The basis $\{\omega_0, \omega_1, \omega_2\}$ of the vector space ${}_{\mathrm{inv}}\Gamma$ is dual to the basis $\{X_0, X_1, X_2\}$ of $\mathcal{T}$. Therefore, by the general theory of left-covariant differential calculi we have

$$\mathrm{d}x = \sum_{j=0}^{2} x_{(1)}\langle X_j, x_{(2)}\rangle \omega_j, \quad x \in \mathcal{A}. \tag{25}$$

From (24) and the relations $\langle X_0, b\rangle = \langle X_2, c\rangle = \langle X_1, a\rangle = 1$ and $\langle X_0, a\rangle = \langle X_0, c\rangle = \langle X_2, a\rangle = \langle X_2, b\rangle = \langle X_1, b\rangle = \langle X_1, c\rangle = 0$ by (11), we obtain

$$\begin{aligned}
&\langle X_0, x_+\rangle = \langle X_2, x_-\rangle = q^{-1},\\
&\langle X_0, x_-\rangle = \langle X_0, y_0\rangle = \langle X_2, x_+\rangle = \langle X_2, y_0\rangle = \langle X_1, x_+\rangle = \langle X_1, x_-\rangle = \langle X_1, y_0\rangle = 0.
\end{aligned}$$

Inserting these facts and Eqs. (19)–(21) into (25) we get

$$\mathrm{d}x_+ = q^{-1}a^2\omega_0 + b^2\omega_2, \quad \mathrm{d}x_- = c^2\omega_0 + qd^2\omega_2, \quad \mathrm{d}y_0 = ca\omega_0 + bd\omega_2. \tag{26}$$

Some lengthy but straightforward computations using formulas (26), (6) and (7) yield the following commutation relations for the FODC Γ_2 on $\mathcal{O}(S_q^2)$:

$$\begin{aligned}
\mathrm{d}x_+x_+ &= x_+\,\mathrm{d}x_+ - q^{-1}\lambda x_+^2\,\mathrm{d}x_0 + q\lambda x_+x_0\,\mathrm{d}x_+,\\
\mathrm{d}x_+x_- &= q^2x_-\,\mathrm{d}x_+ + q\lambda x_+x_-\,\mathrm{d}x_0 - q^{-1}\lambda x_+(x_0-1)\,\mathrm{d}x_-,\\
\mathrm{d}x_+x_0 &= x_0\,\mathrm{d}x_+ + q\lambda x_+(x_0+q^{-2})\,\mathrm{d}x_0 - q^{-1}\lambda\lambda_+^2x_+^2\,\mathrm{d}x_-,\\
\mathrm{d}x_-x_+ &= q^{-2}x_+\,\mathrm{d}x_- - q^{-1}\lambda x_-x_+\,\mathrm{d}x_0 - q\lambda x_-(x_0-1)\,\mathrm{d}x_+,\\
\mathrm{d}x_-x_- &= x_-\,\mathrm{d}x_- + q\lambda x_-^2\,\mathrm{d}x_0 - q^{-1}\lambda x_-x_0\,\mathrm{d}x_-,\\
\mathrm{d}x_-x_0 &= x_0\,\mathrm{d}x_- - q^{-1}\lambda x_-(x_0+q^2)\,\mathrm{d}x_0 + q\lambda\lambda_+^2x_-^2\,\mathrm{d}x_+,\\
\mathrm{d}x_0x_+ &= q^{-2}x_+\,\mathrm{d}x_0 + q^{-1}\lambda x_+(x_0+q^{-2})\,\mathrm{d}x_0 - q^{-1}\lambda(x_0-1)\,\mathrm{d}x_+ - q^3\lambda\lambda_+^2x_+^2\,\mathrm{d}x_-.
\end{aligned}$$

$$\mathrm{d}x_0x_- = q^2x_-\,\mathrm{d}x_0 + q\lambda(x_0-1)\,\mathrm{d}x_- - q\lambda x_-(x_0+q^2)\,\mathrm{d}x_0 + q^3\lambda\lambda_+^2x_-^2\,\mathrm{d}x_+,$$
$$\mathrm{d}x_0x_0 = x_0\,\mathrm{d}x_0 - q^{-1}\lambda\lambda_+^2(x_0-1)x_+\,\mathrm{d}x_- + q\lambda x_0(x_0-1)\,\mathrm{d}x_0.$$

Moreover, from (26) it follows also that

$$x_+\,\mathrm{d}x_- + q^2x_-\,\mathrm{d}x_+ - qx_0\,\mathrm{d}y_0 = 0. \tag{27}$$

Lemma 4. *The* FODC Γ_2 *is the left counter-part of the unique two-dimensional left-covariant *-FODC on* $\mathcal{O}(S_q^2)$ *characterized in* [10].

Proof. The assertion will follow from the first statement of the main theorem in [10]. Hence it suffices to check that Γ_2 fulfills the assumptions made there. Being induced from the left-covariant *-FODC Γ on $\mathcal{O}(SU_q(2))$, Γ_2 is obviously a left covariant *-FODC on $\mathcal{O}(S_q^2)$. The above commutation rules show that the differentials $\mathrm{d}x_+$, $\mathrm{d}x_-$, $\mathrm{d}x_0$ generate Γ_2 as a left $\mathcal{O}(S_q^2)$-module. Thus it remains to verify assumption (7) in [10], Section 1. In the present context this condition means that for arbitrary elements $z_+, z_-, z_0 \in \mathcal{O}(S_q^2)$ an equation

$$z_+\,\mathrm{d}x_+ + z_-\,\mathrm{d}x_- + z_0\,\mathrm{d}y_0 = 0 \tag{28}$$

in Γ_2 is valid if and only if there is an element $z \in \mathcal{O}(S_q^2)$ such that

$$z_+ = q^2zx_-,\quad z_- = zx_+,\quad z_0 = -qzx_0. \tag{29}$$

Clearly, (29) implies (28) because of the relation (27). (By modifying the uniqueness proofs given in [10] or [1] the proof of the converse direction can be avoided in the present case, but we prefer to carry it out here.) Conservely, suppose now that (28) holds. Inserting this into (26) and comparing the coefficients of ω_0 and ω_2 we obtain

$$q^{-1}z_+a^2 + z_-c^2 + z_0ca = 0, \tag{30}$$
$$z_+b^2 + qz_-d^2 + z_0bd = 0. \tag{31}$$

Eqs. (31) q^2ac – (30)db, (31)a^2 – (30)$q^{-3}b^2$ and (30)d^2 – (31)q^3c^2 can be written as

$$q^2z_-x_- - z_+x_+ + q\lambda z_0y_0 = 0, \tag{32}$$
$$z_-(q^{-1}\lambda_+y_0 + q) + z_0x_+ = 0, \tag{33}$$
$$z_+(q\lambda_+y_0 + q^{-1}) + z_0x_- = 0, \tag{34}$$

respectively. Define now an element $z \in \mathcal{O}(S_q^2)$ by

$$z := -\lambda_+^2z_-x_- - z_0(q\lambda y_0 + q^{-1}) = -q^{-2}\lambda_+^2z_+x_+ - z_0(q^{-3}\lambda_+y_0 + q^{-1}), \tag{35}$$

where the second equality follows from (32). From the algebra relations (22) and (23) and the formulas (33) and (34) it then follows that the relations (29) are fulfilled. For instance, let us explain how to get the first equality of (29). First we multiply the second representation of z in (35) by x_- from the right, then we symplify the terms by means of the algebra relations $x_+x_- = q^2y_0^2 + qy_0$ and $y_0x_- = q^2x_-y_0$ and finally we insert the expression of z_0x_- from (34). This in turn yields the desired relation $zx_- = q^{-2}z_+$. The second and third equalities in (29) can be derived in a similar manner from the first expression of z in (35) and formula (33). □

4. Main results

The first main theorem describes all possible admissible commutator representations of the $3D$-calculus on $SU_q(2)$. In order to formulate this result some further preliminaries are needed.

Let π be a *-representation of $\mathcal{A} = \mathcal{O}(SU_q(2))$ as described by the model in Section 3. Suppose that a linear operator T and a symmetric linear operator R on $\mathcal{H}_0$ and a dense linear subspace $\mathcal{D}_0 \subseteq \mathcal{D}(T) \cap \mathcal{D}(T^*) \cap \mathcal{D}(R)$ of $\mathcal{H}_0$ such that $w\mathcal{D}_0 = \mathcal{D}_0$,

$$wTw^*\eta = qT\eta \quad \text{for } \eta \in \mathcal{D}_0, \tag{36}$$

$$w^2Rw^{*2}\eta + q^2R\eta = (1+q^2)wRw^*\eta \quad \text{for } \eta \in \mathcal{D}_0. \tag{37}$$

From the assumptions $w\mathcal{D}_0 = \mathcal{D}_0$ and (36) one easily derives that

$$\begin{aligned} &wT\eta = qTw\eta, \quad wT^*\eta = qT^*w\eta, \quad Tw^*\eta = qw^*T\eta, \\ &\quad T^*w^*\eta = qw^*T^*\eta \quad \text{for } \eta \in \mathcal{D}_0. \end{aligned} \tag{38}$$

Further, assume that there are a symmetric linear operator Q on $\mathcal{G}$ and a dense linear subspace $\mathcal{E}$ of $\mathcal{G}$ such that $v\mathcal{D}_0 = \mathcal{D}_0$ and

$$v^2Qv^{*2}\eta + q^2Q\eta = (1+q^2)vQv^*\eta \quad \text{for } \eta \in \mathcal{D}_0. \tag{39}$$

Let F be a linear operator on the Hilbert space $\mathcal{G} \oplus \mathcal{H}$ which has the dense linear subspace $\mathcal{D}_F := \mathcal{E} \oplus \mathrm{Lin}\{\eta_n : \eta \in \mathcal{D}_0, n \in \mathbb{N}_0\}$ as a core and is defined by

$$F\eta_n = \lambda_nT\eta_{n-1} + w^nRw^{*n}\eta_n + \lambda_{n+1}T^*\eta_{n+1}, \quad \eta \in \mathcal{D}_0, \tag{40}$$

$$F\eta = Q\eta, \quad \eta \in \mathcal{E}. \tag{41}$$

Clearly, F is a symmetric operator.

Theorem 1. *Under the above assumptions, the pair (π, F) is an admissible commutator representation of the $3D$-calculus on $SU_q(2)$. Up to unitary equivalence any admissible commutator representation of the $3D$-calculus is of this form.*

We shall see in Section 5 that by appropriate choice of the above operators T and R one obtains a faithful admissible commutator representation of the $3D$-calculus. In contrast to this the $4D_\pm$-calculus has no faithful admissible commutator representation.

Theorem 2. *Let (π, F) be an admissible commutator representation of the $4D_\pm$-calculus $\Gamma_\pm$. Then the corresponding* *-FODC *homomorphism $\rho : \Gamma_\pm \to \Gamma_{\pi,F}$ passes to a* *homomorphism of the quotient* *-FODC $\Gamma_{\pm,3}$ $\Gamma_{\pi,F}$, *so* (π, F) *becomes a commutator representation of* $\Gamma_{\pm,3}$.

The next theorem shows in particular that none of the three calculi can be given by a spectral triple in the sense of Connes.

Theorem 3. *If* (π, F) *is a commutator representation of the* $3D$*-calculus or the* $4D_{\pm}$*-calculus such that all operators* $\mathrm{d}_{\pi,F}(x)$, $x \in \mathcal{A}$, *are bounded, then we have* $\mathrm{d}_{\pi,F}(x) = 0$ *for all* $x \in \mathcal{A}$.

5. Commutator representations of the $3D$-calculus on $SU_q(2)$

In this section we investigate admissible commutator representations (π, F) of the $3D$-calculus Γ more in detail. Throughout this section we retain the notation of Sections 3 and 4 and suppose that π is a *-representation of $\mathcal{A} = \mathcal{O}(SU_q(2))$ such that $\mathcal{G} = \{0\}$. If not specified otherwise all operator equations containing the operators R, T and F are meant on the domains $\mathcal{D}_0$ and $\mathcal{D}_F = \mathrm{Lin}\{\eta_n; \eta \in \mathcal{D}_0\}$, respectively. Further, we will denote an operator on $\mathcal{H}_0$ and the corresponding diagonal operator on $\mathcal{H} = \oplus_n \mathcal{H}_n$ by the same symbol.

Let us first look at the operator relation (37). It can be rewritten in the form

$$w(wRw^* - R)w^* = q^2(wRw^* - R). \tag{42}$$

Therefore, if R' and R'' satisfies the operator equations

$$wR'w^* = q^2 R' \quad \text{and} \quad wR''w^* = R'', \tag{43}$$

respectively, then $R := R' + R''$ is a solution of Eq. (37). Conversely, suppose that R is a solution of (37) and put

$$R' := (1+q^2)^{-1}(R - wRw^*) \quad \text{and} \quad R'' = (1+q^2)^{-1}(q^2 R + wRw^*).$$

Then R' and R'' satisfy Eq. (43) and we have $R := R' + R''$. From this decomposition and Lemma 6 below it follows in particular that the only *bounded* solutions of (37) are the bounded operators commuting with w. Further, we obtain that

$$R_n \eta_n \equiv w^n R w^{*n} \eta_n = q^{2n} R' \eta_n + R'' \eta_n, \quad \eta \in \mathcal{D}_0. \tag{44}$$

If (π, F) is a commutator representation of Γ with homomorphism $\rho : \Gamma \to \Gamma_{\pi,F}$, we let $\Omega_j = \rho(-\mathrm{i}\omega_j)$ denote the image of the left-invariant 1-form $-\mathrm{i}\omega_j$, $j = 0, 1, 2$. Then we have

$$\Omega_0 = \Omega_{\pi,F}(b), \quad \Omega_2 = \Omega_{\pi,F}(c), \quad \Omega_1 = \Omega_{\pi,F}(a) = -q^{-2}\Omega_{\pi,F}(d).$$

The next theorem gives a reformulation of admissible commutator representations in terms of the representation π. It shows that the operators, $\Omega_0, \Omega_2, \Omega_1$ can be nicely expressed in terms of the operators T and R'. Note that $\pi(c)^{-1}$ is a well-defined bounded operator mapping $\mathcal{D}_F$ into itself, because we assumed that $\mathcal{G} = \{0\}$.

Theorem 4. *Suppose that (π, F) is an admissible commutator representation of the $3D$-calculus and let F be of the form* (40) *with operators T and $R = R' + R''$ satisfying* (36) *and* (37), *respectively. Then we have*

$$\pi(c)T = qT\pi(c) \quad \textit{and} \quad \pi(c)R' = q^2 R'\pi(c), \tag{45}$$

$$R''\pi(x) = \pi(x)R'' \quad \textit{for all } x \in \mathcal{A}, \tag{46}$$

$$F\eta_n = T\pi(a)\eta_n + \pi(c)^n R'\pi(c)^{-n}\eta_n + R''\eta_n + T^*\pi(d)\eta_n, \quad \eta \in \mathcal{D}_0, \tag{47}$$

$$\Omega_0 = \lambda\pi(b)T, \quad \Omega_2 = -\lambda\pi(c)T^*, \quad \Omega_1 = q^{-2}\lambda\pi(bc)R'. \tag{48}$$

Conversely, if R' and R'' are symmetric linear operators and T is a linear operator defined on common dense linear subspace $\mathcal{D}_0 \subseteq \mathcal{D}(T) \cap \mathcal{D}(T^*) \cap \mathcal{D}(R)$ of the Hilbert space $\mathcal{H}_0$ such that (45) *and* (46) *are valid, then the pair (π, F) with F defined by* (47) *is an admissible commutator representation of the $3D$-calculus.*

Proof. Most of the assertions are only reformulations of the conditions occurring in Section 4. Therefore we do not carry out all details of proof. For instance, (36) and (37) are equivalent to Eqs. (45) and (46). Since $R'' = wR''w^*$ as noted above, R'' commutes with $\pi(b)$ and $\pi(c)$ and hence with all representation operators $\pi(x), x \in \mathcal{A}$. Formula (47) follows from (40) and (44).

As a sample, we prove the formula for Ω_0 and compute

$$\begin{aligned}
\Omega_0\eta_n &= (\pi(d)F\pi(b) - q^{-1}\pi(b)F\pi(d))\eta_n \\
&= -q^{n+1}\pi(d)(\lambda_n T w^*\eta_{n-1} + w^n R w^{*n+1}\eta_n + \lambda_{n+1}T^*w^*\eta_{n+1}) \\
&\quad -q^{-1}\pi(b)\lambda_{n+1}(\lambda_{n+1}T\eta_n + w^{n+1}Rw^{*n+1}\eta_{n+1} + \lambda_{n+2}T^*\eta_{n+2}) \\
&= -q^{n+1}(\lambda_n^2 T w^*\eta_n + \lambda_{n+1}w^n R w^{*n+1}\eta_{n+1} + \lambda_{n+1}\lambda_{n+2}T^*w^*\eta_{n+2}) \\
&\quad +q^n\lambda_{n+1}^2 w^*T\eta_n + q^{n+1}\lambda_{n+1}w^nRw^{*n+1}\eta_{n+1} + q^{n+2}\lambda_{n+1}\lambda_{n+2}w^*T^*\eta_{n+2} \\
&= (-q^{n+1}\lambda_n^2 + q^{n-1}\lambda_{n+1}^2)Tw^*\eta_n \\
&= q^n(q^{-1} - q)Tw^*\eta_n = \lambda\pi(b)T\eta_n
\end{aligned}$$

for $\eta \in \mathcal{D}_0$. In a similar manner, one shows that

$$\begin{aligned}
\Omega_2\eta_n &= (-q\pi(c)F\pi(a) + \pi(a)F\pi(c))\eta_n = q^{n-1}(q^{-1} - q)wT^*\eta_n, \\
\Omega_1\eta_n &= (\pi(d)F\pi(a) - q^{-1}\pi(b)F\pi(c) - F)\eta_n = (R_{n-1} - R_n)\eta_n \\
&= w^{n-1}(R - wRw^*)w^{*n-1}\eta_n = (1 - q^2)w^{n-1}R'w^{*n-1}\eta_n \\
&= (1 - q^2)q^{2n-2}R'\eta_n.
\end{aligned}$$

These relations imply the two other formulas of (48). □

By the preceding, for a given *-representation π of $\mathcal{A}$ such that $\mathcal{G} = \{0\}$ the operators F of admissible pairs (π, F) are parametrized by the three operators T, R' and R'' on the Hilbert space $\mathcal{H}_0$ satisfying the relations

$$wTw^* = qT, \quad wR'w^* = q^2R' \quad \text{and} \quad wR''w = R''. \tag{49}$$

It is now easy to construct admissible pairs (π, F). We shall do this for the faithful *-representation π of the *-algebra $\mathcal{A}$ given in [14]. In this case w is the backward shift on the Hilbert space $\mathcal{H}_0 = l_2(\mathbb{Z})$. That is, if we identify $\mathcal{H}$ with $l_2(\mathbb{N}_0 \times \mathbb{Z})$ and denote by $\{e_{nk}; n \in \mathbb{N}_0, k \in \mathbb{Z}\}$ the standard orthonormal basis of $l_2(\mathbb{N}_0 \times \mathbb{Z})$, then the operators $w, \pi(a)$ and $\pi(c)$ act as

$$we_{nk} = e_{n,k-1}, \quad \pi(a)e_{nk} = \lambda_n e_{n-1,k}, \quad \pi(c)e_{nk} = q^n e_{n,k-1}. \tag{50}$$

Define linear operators T and R' on the domain $\mathcal{D}_F := \mathrm{Lin}\{e_{nk}; n \in \mathbb{N}_0, k \in \mathbb{Z}\}$ by

$$Te_{nk} = q^k e_{n,k-1}, \quad R'e_{nk} = q^{2k} e_{nk}. \tag{51}$$

Let R'' by a symmetric linear operator on $\mathcal{D}_F$ such that $wR''w^* = R''$. The conditions (49) are obviously fulfilled. By (44), (51) and (40), the action of the operator F on the basis vectors e_{nk} is given by

$$Fe_{nk} = \lambda_n q^k e_{n-1,k-1} + q^{2n+2k} e_{nk} + \lambda_{n+1} q^{k+1} e_{n+1,k+1} + R''e_{nk}.$$

Then the pair (π, F) *is an admissible commutator representation of the* $3D$*-calculus* Γ *on* $SU_q(2)$. For instance, one may take R'' of the form $R''e_{nk} = \sum_r \alpha_r e_{n,k-r}$, where $(\alpha_r; r \in \mathbb{Z})$ is a real sequence such that $\alpha_r = 0$ for $|r| \geq r_0$. In this case it is straightforward to prove that then the corresponding FODC homomorphism $\rho : \Gamma \to \Gamma_{\pi,F}$ is *faithful*. (Indeed, using the vector space basis $\{a^n b^m c^r, b^m c^r d^s; m, n, s \in N_0, n \in \mathbb{N}\}$ of $\mathcal{A}$ and the formulas (48) one verifies that any relation $\pi(x_0)\Omega_0 + \pi(x_1)\Omega_1 + \pi(x_2)\Omega_2 = 0$ with $x_0, x_1, x_2 \in \mathcal{A}$ implies that $x_0 = x_1 = x_2 = 0$.)

Note that the operators $\mathrm{d}_{\pi,F}(x) = [\mathrm{i}F, \pi(x)]$, $x \in \mathcal{A}$, of the FODC $\Gamma_{\pi,F}$ are unbounded. This stems from the fact that T and R'' and hence the basis elements Ω_j, $j = 0, 1, 2$, of the vector space of left-invariant 1-forms of $\Gamma_{\pi,F}$ are unbounded operators. The reason are the sequences (q^k) resp. (q^{2k}) in (51) as $k \to -\infty$, so this unboundedness is rather well controlled.

Commutator representations can be used to construct extensions of *-FODC to larger algebras. We explain this for the $3D$-calculus. It is clear that the set $\mathcal{S} := \{b^n c^m; n, m \in \mathbb{N}_0\}$ is a left Ore subset of the algebra $\mathcal{A}$ (that is, for any $(s, x) \in \mathcal{S} \times \mathcal{A}$ there exists $(t, y) \in \mathcal{S} \times \mathcal{A}$ such that $ys = tx$). Moreover, the algebra $\mathcal{A}$ has no zero divisors. Therefore, as it is well known in ring theory, there exists a *-algebra $\tilde{\mathcal{A}}$ which contains $\mathcal{A}$ as a *-subalgebra such that the elements of $\mathcal{S}$ are invertible and $\tilde{\mathcal{A}}$ is generated by $\mathcal{A}$ and the inverses of $\mathcal{S}$. Since the Hilbert space $\mathcal{G}$ is zero, the *-representation π of $\mathcal{A}$ extends uniquely to a *-representation $\tilde{\pi}$ of the *-algebra $\tilde{\mathcal{A}}$. Hence $(\Gamma_{\tilde{\pi},F}, \mathrm{d}_{\tilde{\pi},F})$ is a *-FODC of the *-algebra $\tilde{\mathcal{A}}$. If we take a faithful commutator representation (π, F) of the $3D$-calculus, then we obtain an extension of the $3D$-calculus to the larger *-algebra $\tilde{\mathcal{A}}$ in this manner.

For the study of harmonic analysis and metric noncommutative geometry on $SU_q(2)$ it is more important to work with the direct sum $\pi_{\rm reg}$ of $\mathbb{N}_0$ copies of the *-representation π. That is, we take the Hilbert space $\mathcal{H}_{\rm reg} = l_2(\mathbb{N}_0 \times \mathbb{Z} \times \mathbb{N}_0)$ with standard orthonormal basis $\{e_{nkl}; n, l, \in \mathbb{N}_0, k \in \mathbb{Z}\}$ and let the operators $\pi_{\rm reg}(x), x \in \mathcal{A}$, and w act on the first two indices as stated above. Then $\pi_{\rm reg}$ is just the GNS representation of $\mathcal{A}$ associated with the Haar state h of the compact quantum group algebra $\mathcal{A} = \mathcal{O}(SU_q(2))$. Indeed, if φ_h denotes the vector

$$\varphi_h := (1-q^2)^{-1/2} \sum_{n=0}^{\infty} q^n e_{n0n}, \tag{52}$$

then it follows at once from the explicit formulas for the Haar state [6,8,13,14] that

$$h(x) = \langle \pi_{\rm reg}(x)\varphi_h, \varphi_h \rangle, \quad x \in \mathcal{A}.$$

Let α and β be positive reals. Define operators T, R' and R'' on the span of basis vectors e_{nkl} by

$$\begin{aligned} Te_{nkl} &= \alpha(1+q^2)^{1/2} q^k e_{n,k-1,l-1}, \qquad R'e_{nk} = \beta q^2 (1+q^2+q^4)^{1/2} q^{2k} e_{knl}, \\ R'' &= 0. \end{aligned} \tag{53}$$

Let $F_{\rm reg}$ denote the corresponding operator given by (47). Then the pair $(\pi_{\rm reg}, F_{\rm reg})$ is another admissible commutator representation of the $3D$-calculus. It is natural to use the state vector φ_h to define a scalar product on the 1-forms of the $3D$-calculus Γ by

$$\langle \omega, \omega' \rangle := \langle \rho(\omega)\varphi_h, \rho(\omega')\varphi_h \rangle, \quad \omega, \omega' \in \Gamma, \tag{54}$$

where $\rho : \Gamma \to \Gamma_{\pi_{\rm reg}, F_{\rm reg}}$ is the corresponding *-FODC homomorphism. Using the formulas (52)–(54) we compute

$$\langle \omega_0, \omega_0 \rangle = \langle \omega_2, \omega_2 \rangle = \alpha^2, \quad \langle \omega_1, \omega_1 \rangle = \beta^2 \quad \text{and} \quad \langle \omega_k, \omega_l \rangle = 0 \quad \text{if } k \neq l.$$

6. Commutator representations of the two-dimensional calculus on S_q^2

By Lemma 4 we have shown that the $3D$-calculus Γ on the quantum group $SU_q(2)$ induces the two-dimensional calculus Γ_2 on the quantum 2-sphere S_q^2. Thus any commutator representation of the $3D$-calculus gives obviously a commutator representation of the *-subalgebra $\mathcal{O}(S_q^2)$ of $\mathcal{O}(SU_q(2))$. In this brief section we shall make this more explicit.

Let (π, F) be an admissible commutator representation of the $3D$-calculus as described in Section 4, where the *-representation π is as in Section 3.1. Using the condition $wTw^* = qT$ by (36) and formulas (3) and (4) we compute the differentials of the generators x_+, x_-, y_0 and obtain

$$\begin{aligned} \mathrm{d}_{\pi,F}(x_+) &= \mathrm{i}q^{-1}\lambda\pi(a)T\pi(x_+) - \mathrm{i}\lambda\pi(b)T^*\pi(y_0), \\ \mathrm{d}_{\pi,F}(x_-) &= -\mathrm{i}q\lambda\pi(d)T\pi(x_-) + \mathrm{i}\lambda\pi(c)T\pi(y_0), \\ \mathrm{d}_{\pi,F}(y_0) &= \mathrm{i}q^{-1}\lambda\pi(a)T\pi(y_0) - \mathrm{i}q\lambda\pi(d)T\pi(y_0) \\ &= \mathrm{i}\lambda\pi(c)T\pi(x_+) - \mathrm{i}\lambda\pi(b)T^*\pi(x_-). \end{aligned}$$

These formulas describe the corresponding commutator representation of the *-FODC Γ_2 of $\mathcal{O}(S_q^2)$. In particular we see that the operator R does not occur in these formulas and that $\mathrm{d}_{\pi,F}(x) = 0$ on the subspace $\mathcal{G}$ for all $x \in \mathcal{O}(S_q^2)$.

Conversely, let π be a *-representation of $\mathcal{O}(SU_q(2))$ on the Hilbert space $\mathcal{H} = \oplus_n \mathcal{H}_n$ as in Section 3.1 and let T be a linear operator on the Hilbert space $\mathcal{H}_0$. If there exists a dense linear subspace $\mathcal{D}_0 \subseteq \mathcal{D}(T) \cap \mathcal{D}(T^*)$ of $\mathcal{H}_0$ such that $w\mathcal{D}_0 = \mathcal{D}_0$ and $wTw^*\eta = qT\eta$ for $\eta \in \mathcal{D}_0$, then the above formulas describe a commutator representation of the *-FODC Γ_2 of $\mathcal{O}(S_q^2)$. Examples can be constructed similarly as in the case of the $3D$-calculus.

Using the *-FODC $(\Gamma_{\tilde{\pi},F}\,\mathrm{d}_{\tilde{\pi},F})$ of the Ore extension $\tilde{\mathcal{A}}$ of the *-algebra $\mathcal{A}$ defined in Section 5, the operators T and T^* can be expressed by the formulas

$$T = \mathrm{i}\lambda^{-1} b\, \mathrm{d}_{\tilde{\pi},F}(db^{-1}) \quad \text{and} \quad T^* = -\mathrm{i}\lambda^{-1} c\, \mathrm{d}_{\tilde{\pi},F}(ac^{-1}). \tag{55}$$

Remark 2. *Let $z := ac^{-1}$. In the *-algebra $\tilde{\mathcal{A}}$ we then have $z^* = -db^{-1}$ and*

$$z^* z - q^2 z z^* = q^2 - 1. \tag{56}$$

The *-subalgebra $\mathcal{Z}$ of $\tilde{\mathcal{A}}$ generated by the element $z := ac^{-1}$ is just the abstract *-algebra with a single generator z and defining relation (56). It is well known that this *-algebra $\mathcal{Z}$ has a *-FODC with commutation relations

$$\mathrm{d}z \cdot z = q^2 z\, \mathrm{d}z, \quad \mathrm{d}z \cdot z^* = q^{-2} z^*\, \mathrm{d}z,$$
$$\mathrm{d}z^* \cdot z = q^2 z\, \mathrm{d}z^*, \quad \mathrm{d}z^* \cdot z^* = q^{-2} z^*\, \mathrm{d}z^*.$$

These relations can be found (for instance) in [11]. As a byproduct of the preceding consideration we obtain a commutator representation (π, F) of this *-FODC, where π denotes the restriction to $\mathcal{Z}$ of the above *-representation $\tilde{\pi}$ of $\tilde{\mathcal{A}}$ and F is the operator given by (40) with $R = 0$ and $wTw^* = qT$. That is, we have

$$\pi(z)\eta_n = q^{-n} w^* \lambda_n \eta_{n-1}, \quad \pi(z^*)\eta_n = q^{-n-1} w \lambda_{n+1} \eta_{n+1},$$
$$F\eta_n = \lambda_n T \eta_{n-1} + \lambda_{n+1} T^* \eta_{n+1}.$$

From formula (55) we see that then the operators of the differentials $\mathrm{d}_{\pi,F}(z)$ and $\mathrm{d}_{\pi,F}(z^*)$ act as

$$\mathrm{d}_{\pi,F}(z)\eta_n = \mathrm{i}\lambda q^{-n} w^* T^* \eta_n, \quad \mathrm{d}_{\pi,F}(z^*)\eta_n = -\mathrm{i}\lambda q^{n-1} w T \eta_n.$$

7. Proofs of Theorems 1–3

Let us begin with some notation. For simplicity we write x for the representation operator $\pi(x)$ of an algebra element $x \in \mathcal{A}$. Further, we shall omit the symbols $\lceil \mathcal{D}_0$ and $\lceil \mathcal{D}$ denoting the restrictions of the operators to $\mathcal{D}_0$ and $\mathcal{D}$, respectively. Moreover, we write simply $\Omega(\cdot)$ instead of $\Omega_{\pi,F}(\cdot)$.

First let Γ be the $3D$-calculus on $SU_q(2)$. We want to prove that the pair (π, F) defined at the beginning of Section 5 is indeed a commutator representation of Γ. By Lemma 3, it suffices to show that $\Omega(x) = 0$ for the six generators x of the right ideal $\mathcal{R}_\Gamma$ listed by formula (5). Computing the corresponding expressions of $\Omega(x)$ by using formula (1), we obtain the relations

$$\Omega(q^2b^2) \equiv q^2d^2Fb^2 + b^2Fd^2 - (q^2+1)bdFdb = 0, \tag{57}$$

$$\Omega(c^2) \equiv q^2c^2Fa^2 + a^2Fc^2 - (q^2+1)acFca = 0, \tag{58}$$

$$\Omega(qbc) \equiv -q^2cdFba + (q^2+1)bcFbc - abFdc + qFbc + qbcF = 0, \tag{59}$$

$$\Omega(q^2(a-1)b) \equiv q^2d^2Fab - (q^2+1)bdFbc + b^2Fcd - q^2dFb$$
$$-qbdF + qbFd = 0, \tag{60}$$

$$\Omega((a-1)c) \equiv -qcdFa^2 + (q^2+1)bcFca - baFc^2$$
$$+Fac + qcFa - aFc = 0, \tag{61}$$

$$\Omega((q^2a + d - q^2 - 1)) \equiv q^2dFa + aFd - qbFc - qcFb$$
$$-(1+q^2)F = 0. \tag{62}$$

The relations (57) and (60) follow from (58) and (61), respectively, by applying the adjoint operation and using Eq. (2) and the fact that the operator F is symmetric. Therefore it is sufficient to check (58), (59), (61) and (62). We omit these boring straightforward computations. In the course of these verifications formulas (3)–(4) and (40)–(41) for the definition of the representation π and of the operator F and the relations (36)–(39) are essentially used. Thus, (π, F) is indeed a commutator representation of Γ. The admissibility of (π, F) is obvious from its definition. This completes the proof of the first assertion of Theorem 1.

The next part of this section is devoted to the proofs of the second assertion of Theorem 1 and of Theorem 2. For this we suppose that Γ is either the $3D$-calculus, the $4D_+$-calculus or the $4D_-$-calculus on $SU_q(2)$ and that (π, F) is an arbitrary admissible commutator representation of Γ. Let $\mathcal{D}_0$, $\mathcal{E}$ and $\mathcal{D}$ be corresponding subspaces.

Lemma 5.
(i) If Γ is the $3D$-calculus, then we have

$$q^2c^2Fa^2 + a^2Fc^2 - (q^2+1)acFca = 0, \tag{63}$$

$$qc^2Fa + qaFc^2 - q^2cFac - caFc = 0, \tag{64}$$

$$-q^2cdFba + (q^2+1)bcFbc - abFdc + qFbc + qbcF = 0, \tag{65}$$

$$q^2dFa + aFd - qbFc - qcFb - (q^2+1)F = 0. \tag{66}$$

(ii) If Γ is the $4D_\pm$-calculus, then we have Eq. (63) and

$$aFc^2 - \epsilon caFc - q^2cFca + \epsilon qc^2Fa = 0, \tag{67}$$

$$-q^2dcFa^2 - baFc^2 - q^3c^2Fab - qa^2Fcd + (q^2+1)bcFca$$
$$+q(q^2+1)acFbc + qFca + q^3caF = 0. \tag{68}$$

Proof. (i): By (5), the right ideal $\mathcal{R}_\Gamma$ associated with the $3D$-calculus contains the elements $c^2, bc, q^2a + d - (q^2+1)$ and $(a-1)c$. Therefore, by Lemma 2 we have $\Omega_{\pi,F}(c^2) = 0$,

$\Omega_{\pi,F}(bc) = 0$, $\Omega_{\pi,F}(q^2a + d - (q^2+1)) = 0$ and $\Omega_{\pi,F}((a-1)c) = 0$. Computing these expressions by using (1) leads to Eqs. (63), (65), (66) and

$$(q^2+1)bcFca - qcdFa^2 - baFc^2 + Fac + qcFa - aFc = 0, \tag{69}$$

respectively. It remains to derive Eq. (64). If we subtract Eq. $a(66)c$ from $q^2(69)$, we obtain

$$q^2bcFca + qacFbc - q^3cdFa^2 - a^2Fdc + q^3cFa + aFc = 0. \tag{70}$$

Adding (70) and $qc(66)a$ yields the equation

$$(q^2+1)caFbc - q^2c^2Fba - a^2Fdc + qcaF + aFc - qcFa = 0. \tag{71}$$

Inserting the relation $a^2Fc^2 = -q^2c^2Fa^2 + (q^2+1)acFca$ by (63) into (71)c we finally get Eq. (64) as asserted.

(ii) Since the three elements $x = c^2, qc(a-d), z_{\pm}c$ belong to the right ideal associated with the $4D_{\pm}$-calculus (see (18)), the corresponding operators $\Omega_{\pi,F}(x)$ are zero by Lemma 2. This leads to Eqs. (63), (68) and

$$\begin{aligned}&-q^3cdFa^2 + q^2adFac + q^2bcFca - qabFc^2 + q^2c^2Fba - qacFbc\\ &\quad -qcaFda + a^2Fdc + \epsilon(q^4+1)cFa - \epsilon(q^3+q^{-1})aFc = 0,\end{aligned} \tag{72}$$

respectively. We still have to verify Eq. (67). Dividing (68) $- q^2$(72) by q^4+1 and simplifying the terms by using the commutation rules of the matrix entires a, b, c, d, we obtain the equation

$$(q^2+1)bcFca - baFc^2 - q^2dcFa^2 + qFca - q\epsilon aFc + q^2\epsilon cFa = 0. \tag{73}$$

If we substitute $q^2c^2Fa^2 = (q^2+1)acFca - a^2Fc^2$ (by (63)) into $q^{-1}c$ (73), then Eq. (67) follows. □

Now we make use of the structure of the *-representation π and of the admissibility of the pair (π, F). We freely use the notation established above. Let $\mathcal{D}^n$ be the direct sum of domains $\mathcal{D}_k = \{\eta_k : \eta \in \mathcal{D}_0\}, k = 0, \ldots, n$, and let $\mathcal{H}^n$ denote the direct sum of subspaces $\mathcal{H}_k, k = 0, \ldots, n$, of $\mathcal{H}$. Using essentially relation (63) and the fact that ker $a^k = \mathcal{H}_{k-1}$, a straightforward induction argument shows that the operator F maps each space $\mathcal{D}^n$ into $\mathcal{H}^{n+1}$. This in turn implies that F maps the subspace $\mathcal{D} = \mathrm{Lin}\{\mathcal{D}_n; n \in \mathbb{N}_0\}$ into $\mathcal{H} = \oplus_n \mathcal{H}_n$. Since F is symmetric, it follows that F maps the domain $\mathcal{E}$ into $\mathcal{G}$. By assumption, $\mathcal{E} \oplus \mathcal{D}$ is a core for F. Therefore, the operator F and hence all operators $\Omega_{\pi,F}(x)$, $x \in \mathcal{A}$, leave the spaces $\mathcal{G}$ and $\mathcal{H}$ invariant. Using once more the facts that the operator F is symmetric and that F maps $\mathcal{D}^n$ into $\mathcal{H}^{n+1}$ it follows that the restriction of the operator F to the dense linear subspace $\mathcal{D}$ of $\mathcal{H}$ is of the form

$$F\eta_n = T_n\eta_{n-1} + R_n\eta_n + T_{n+1}^*\eta_{n+1}, \quad \eta \in \mathcal{D}_0. \tag{74}$$

Here T_n and R_n, $n \in \mathbb{N}_0$, are (possible unbounded) linear operators on the Hilbert space $\mathcal{H}_0$ such that the domains of T_n, R_n and T_n^* contain $\mathcal{D}_0$ and R_n is symmetric. Formula (74) will be essentially used in the sequel. For $n \in \mathbb{N}$, we set

$$E_n := R_n - wR_{n-1}w^*. \tag{75}$$

Inserting the formulas (4) and (74) for the action of the operators a, c and F into (63) and comparing the expressions occurring in the $(n-2)$th, $(n-1)$th and nth components, we obtain the recurrence relations

$$\lambda_{n+1}\lambda_n w^2 T_{n-1} + q^4\lambda_n\lambda_{n-1}T_{n+1}w^2 = (q+q^3)\lambda_{n+1}\lambda_{n-1}wT_n w, \tag{76}$$

$$w^2 R_{n-1} + q^2 R_{n+1}w^2 = (q^2+1)wR_n w, \tag{77}$$

$$\lambda_{n+1}\lambda_n w^2 T_n^* + \lambda_{n+2}\lambda_{n+1}T_{n+2}^* w^2 = (q+q^{-1})\lambda_{n+1}^2 wT_{n+1}^* w, \tag{78}$$

respectively. Applying first the adjoint operation to (78), multiplying then by w^{-2} from the left and from the right, dividing by λ_{n+1} and replacing finally n by $n-1$, we get

$$\lambda_{n-1}w^2 T_{n-1} + \lambda_{n+1}T_{n+1}w^2 = (q+q^{-1})\lambda_n wT_n w. \tag{79}$$

Eq. $\lambda_{n-1}(76) - \lambda_n\lambda_{n+1}(79)$ reads as

$$\begin{aligned}&(q^4\lambda_n\lambda_{n-1}^2 - \lambda_n\lambda_{n+1}^2)T_{n+1}w^2\\ &\quad = (q+q^3)\lambda_{n-1}^2\lambda_{n+1}((q+q^3)\lambda_{n-1}^2\lambda_{n+1} - (q+q^{-1})\lambda_n^2\lambda_{n+1})wT_n w.\end{aligned}$$

Since $\lambda_k^2 = 1-q^{2k}$ by definition, the latter yields $\lambda_n T_{n+1}w^2 = q^{-1}\lambda_{n+1}wT_n w$ and so

$$q\lambda_n T_{n+1} = \lambda_{n+1}wT_n w^*. \tag{80}$$

Note that the preceding formulas (74), (77) and (80) are valid for both the $3D$-calculus and the $4D_\pm$-calculus, because they were derived only from Eq. (63) and this equation holds for all three calculi according to Lemma 5.

In order to continue the proof we first specify to the $3D$-calculus. Then, by Lemma 5(i), we have also Eq. (64). Inserting now (74) into (64) and comparing the $(n-1)$th and nth components, we get the relations

$$q^2 R_n w^2 + w^2 R_{n-1} = q^2 wR_{n-1}w + wR_n w, \tag{81}$$

$$\lambda_{n+1}T_{n+1}^* w^2 + \lambda_n w^2 T_n^* = q\lambda_n wT_n^* w + q^{-1}\lambda_{n+1}wT_{n+1}^* w, \tag{82}$$

respectively. Multiplying (82) by w^* from the right, replacing n by $n-1$, passing to the adjoint operators and finally applying formula (80), we derive

$$\lambda_{n+1}T_n = \lambda_n T_{n+1}. \tag{83}$$

Comparing (80) and (83) we conclude that $T_n = \lambda_n\lambda_1^{-1}T_1$ and $wT_n w^* = qT_n$. That is, setting $T := \lambda_1^{-1}T_1$, we have

$$T_n = \lambda_n T \quad \text{and} \quad wTw^*\eta = qT\eta, \quad \eta \in \mathcal{D}_0. \tag{84}$$

Next we investigate the diagonal terms R_n of the operator F. First we note that in terms of the operators E_n defined by (75) the Eqs. (77) and (81) are reformulated as

$$q^2 E_{n+1} = wE_n w^* \quad \text{and} \quad q^2 E_n = wE_n w^*, \tag{85}$$

respectively. In particular, we have $E_{n+1} = E_n$ for all n. If we compare the nth components in (65), we get the relation

$$q^{n+2}\lambda_n^2 wR_{n-1}w^* + (q^2+1)q^{2n+2}R_n \\ +q^{n+2}\lambda_{n+1}^2 w^*R_{n+1}w - 2q^{n+2}R_n = 0. \tag{86}$$

Putting the relations (85) into (86) we derive that $E_n = 0$ for all n. Setting $R := R_0$, the latter means that

$$R_n = wR_{n-1}w^* = w^n R w^{*n}. \tag{87}$$

Further, comparing the nth components in (66), we find that

$$q^2\lambda_n^2 R_{n-1} + \lambda_{n+1}^2 R_{n+1} + q^{2n+2}w^*R_n w + q^{2n+2}wR_n w^* - (q^2+1)R_n = 0. \tag{88}$$

Inserting the relation $E_n = 0$ into (88), we obtain $R_{n+1} + q^2 R_{n-1} - (q^2+1)R_n = 0$. Because of (87), this means that

$$w^2 R w^{*2} + q^2 R = (1+q^2)wRw^*. \tag{89}$$

Finally, the restriction of the operator F to the subspace $\mathcal{E}$ of the Hilbert space $\mathcal{G}$ is a symmetric linear operator, say Q. Since $b = c = 0, a = v$ and $d = v^*$ on $\mathcal{G}$ by (3), Eq. (66) reads as

$$v^2 Q v^{*2} + q^2 Q = (1+q^2)vQv^*. \tag{90}$$

Summarizing the preceding, formulas (74), (84), (87), (89) and (90) show that the operator F has the required form. This completes the proof of the second assertion of Theorem 1.

Now we turn to the $4D_\pm$-calculus and prove Theorem 2. To begin with, we compute the $(n-1)$th and the nth components of the expressions in Eq. (67). Comparing coefficients we derive

$$R_n w^2 - q^{-1}\epsilon wR_n w - wR_{n-1}w + q^{-1}\epsilon w^2 R_{n-1} = 0, \tag{91}$$

$$\lambda_{n+1}T_{n+1}^* w^2 - \epsilon\lambda_{n+1}wT_{n+1}^* w - q\lambda_n wT_n^* w + q\epsilon\lambda_n w^2 T_n^* = 0. \tag{92}$$

Applying the adjoint operation to $(92)w^{*2}$, we get

$$\lambda_{n+1}T_{n+1} - \epsilon\lambda_{n+1}wT_{n+1}w^* - q\lambda_n wT_n w^* + q\epsilon w^2 T_n w^{*2} = 0. \tag{93}$$

Recall that formula (80) holds also for the $4D_\pm$-calculus, because it was derived from formula (63). Inserting (80) into (93), we derive that

$$T_{n+1} = \epsilon wT_{n+1}w^*. \tag{94}$$

Combining the latter with (80), we get

$$q\lambda_n T_{n+1} = \epsilon\lambda_{n+1}T_n. \tag{95}$$

Next we use Eq. (68) which holds by Lemma 5(ii). Computing the $(n-1)$th components of (68), we obtain the relation

$$-q^n\lambda_n\lambda_{n-1}^2 wR_{n-2} + q^{3n}\lambda_n w^* R_n w^2 + q^{3n+2}\lambda_n w^2 R_{n-1} w^*$$
$$-q^{n+2}\lambda_{n+1}^2\lambda_n R_{n+1} w - (q^2+1)q^{3n-2}\lambda_n R_{n-1} w$$
$$-(q^2+1)q^{3n+2}\lambda_n w R_n + q^n\lambda_n R_{n-1} w + q^{n+2}\lambda_n w R_n = 0. \tag{96}$$

In terms of the operator E_n, formulas (77) and (91) can be written as

$$q^2 E_{n+1} = wE_n w^* \quad \text{and} \quad E_n = \epsilon q^{-1} w E_n w^*, \tag{97}$$

respectively. Inserting these formulas into $(96)w^*$, a lengthy computation shows that $E_n = 0$. That is, setting $R := R_0$, we have

$$R_n = wR_{n-1}w^* = w^n R w^{*2} \quad \text{for } n \in \mathbb{N}_0. \tag{98}$$

Using formulas (72), (73) and (98) established above, we compute

$$\Omega(a)\eta_n = (\epsilon q - 1)T_n\eta_{n-1} + (R_{n-1} - R_n)\eta_n + (\epsilon q - 1)T_{n+1}^*\eta_{n+1}, \tag{99}$$
$$\Omega(b)\eta_n = \lambda q^n \lambda_n^{-1} T_n w^* \eta_n \equiv \lambda q^{-1}\lambda_n^{-1} T_n b \eta_n, \tag{100}$$
$$\Omega(c)\eta_n = -\lambda q^n \lambda_n^{-1} w T_n^* \eta_n \equiv -\lambda\lambda_n^{-1} c T_n^* \eta_n, \tag{101}$$
$$\Omega(d)\eta_n = (\epsilon q^{-1} - 1)T_n\eta_{n-1} + (R_{n+1} - R_n)\eta_n + (\epsilon q^{-1} - 1)T_{n+1}^*\eta_{n+1}, \tag{102}$$

for $\eta \in \mathcal{D}_0$ and $n \in \mathbb{N}$. In particular, we get

$$\Omega(a + \epsilon q d)\eta_n = (\epsilon q R_{n+1} - (\epsilon q + 1)R_n + R_{n-1})\eta_n. \tag{103}$$

Put $\Omega_j = \rho(-\mathrm{i}\omega_j)$. Since ρ is bimodule homomorphism, the commutation relations between the 1-forms ω_j and the generators a, b, c, d remain valid if ω_j is replaced by Ω_j. In particular, the relation $\omega_2 a = \epsilon a \omega_2 + \epsilon\lambda^2 q^{-1} b\omega_4$ yields

$$\Omega_2 a = \epsilon a \Omega_2 + \epsilon\lambda^2 q^{-1} b \Omega_4. \tag{104}$$

Since $\Omega(a + \epsilon q d) = (1 - q^2)(\epsilon q^{-3} - 1)\Omega_4$ by Ω_4 by (14) and (15), it follows from (103) that $\Omega_4 : \mathcal{H}_n \to \mathcal{H}_n$ and so $b\Omega_4 : \mathcal{H}_n \to \mathcal{H}_n$. Further, by (100) the operators $\Omega_2 a$ and $\epsilon a \Omega_2$ map $\mathcal{H}_n$ into $\mathcal{H}_{n-1}$. Because of (104) this implies that $\Omega_4 = 0$ on each space $\mathcal{H}_n$ and so on $\mathcal{H}$. From the relation $\omega_2 c = \epsilon c \omega_2 + \epsilon\lambda^2 q^{-1} d\omega_4$ we get $\Omega_2 c = \epsilon c \Omega_2 + \epsilon\lambda^2 q^{-1} d\Omega_4$. Since $c = 0$ and d is unitary on $\mathcal{G}$, this implies that $\Omega_4 = 0$ on $\mathcal{G}$.

Thus we have shown that $\Omega_4 = 0$ on $\mathcal{G} \oplus \mathcal{H}$ which yields that $\rho(\omega_{\Gamma_\pm}(a + \epsilon q d)) = 0$. Therefore, by Lemma 3 the FODC homomorphism ρ passes to the quotient FODC $\Gamma_{\pm,3}$. This completes the proof of Theorem 2.

Remark 3. *The assertions of Theorems 1 and 2 could be also derived from the commutation relations and the involution properties of the calculi thus avoiding the use of the right ideals. We preferred to give the above proof because it emphasizes the role of the corresponding right ideals and it needs only very few generators of the right ideals.*

Remark 4. *By adding a few lines to the preceding arguments one gets a complete description of all admissible commutator representations of the quotient* *-FODC $\Gamma_{\pm,3}$. *In this* *remark we briefly derive this result. Since* $\Omega_4 = 0$ *and hence* $\Omega(a + \epsilon q d) = 0$, *it follows from* (98), (99) *and* (102) *that*

$$\omega^2 R \omega^{*2} + \epsilon q^{-1} R = (1 + \epsilon q^{-1}) w R w^* \quad \text{on } \mathcal{D}_0. \tag{105}$$

Set $T := \lambda_1^{-1} T_1$. *From* (94) *and* (95) *we get* $T_n = (\epsilon q)^{1-n} \lambda_n T$ *and*

$$w T w^* = \epsilon T \quad \text{on } \mathcal{D}_0. \tag{106}$$

Let Q *denote the restriction of* F *to the domain* $\mathcal{E}$ *in the subspace* $\mathcal{G}$. *Since* $b = 0$ *on* $\mathcal{G}$, *we have* $\Omega(b) \equiv dFb - q^{-1} bFd = 0$ *and so* $\Omega_2 = 0$ *on* $\mathcal{G}$. *Hence the relation* $\Omega_1 d = \epsilon q^{-1} d \Omega_1 + \epsilon c \Omega_2$ *implies that* $\Omega_1 v^* = \epsilon q^{-1} v^* \Omega_1$ *on* $\mathcal{G}$. *Because* $\Omega_4 = 0$ *as shown in the above proof, we have* $\Omega(a) = (\epsilon q - 1)\Omega_1$ *by* (14). *On* $\mathcal{G}$ *we have* $\Omega(a) \equiv dFa - q^{-1} bFc - F = v^* Q v - Q$. *Inserting the latter into the relation* $\Omega(a) v^* = \epsilon q^{-1} v^* \Omega(a)$, *we finally obtain that*

$$v^2 Q v^{*2} + \epsilon q^{-1} Q = (1 + \epsilon q^{-1}) v Q v^* \quad \text{on } \mathcal{E}. \tag{107}$$

The symmetric operator F *now acts as*

$$F\eta_n = (\epsilon q)^{1-n} \lambda_n T \eta_{n-1} + w^n R w^{*n} \eta_n + (\epsilon q)^{-n} \lambda_{n+1} T^* \eta_{n+1}, \quad \eta \in \mathcal{D}_0, \tag{108a}$$

and

$$F\eta = Q\eta, \quad \eta \in \mathcal{E}. \tag{108b}$$

Conversely, let T *be a linear operator and* R *a symmetric linear operator on a dense domain* $\mathcal{D}_0$ *of* $\mathcal{H}$ *and let* Q *be a symmetric linear operator on a dense domain* $\mathcal{E}$ *of* $\mathcal{G}$ *such that* $\mathcal{D}_0 \subseteq \mathcal{D}(T^*)$, $w\mathcal{D}_0 = \mathcal{D}_0$ *and* $v\mathcal{E} = \mathcal{E}$. *If the relations* (105)–(107) *are fulfilled, then the pair* (π, F) *with* F *defined by* (108a) *and* (108b) *is an admissible commutator representation of the* $*$*-FODC* $\Gamma_{\pm,3}$.

Now let us turn to the proof of Theorem 3. The key of the proof is the following simple lemma.

Lemma 6. *Let* u *be a unitary operator and let* A *be a bounded linear operator on a Hilbert space. If* $uAu^* = \alpha A$ *for some* $\alpha \in \mathbb{R}$, $|\alpha| \neq 1$, *then* $A = 0$.

Proof. Then we also have $uA^*u^* = \alpha A^*$. So we may assume that A is self-adjoint. The relation $uAu^* = \alpha A$ implies that the spectrum of A is invariant under multiplication by α and α^{-1}. Since $|\alpha| \neq 1$ and A is bounded, this is only possible if $A = 0$. □

We give the proof for the $4D_\pm$-calculus. By assumption all operators $\mathrm{d}_{\pi,F}(x)$, $x \in \mathcal{A}$, are bounded. Hence the four operators $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ are also bounded. Recall that $\Omega_1^* = \Omega_1$, $\Omega_2^* = \Omega_3$ and $\Omega_4^* = \Omega_4$ by (17). From these facts and the commutation relations of the $4D_\pm$-calculus it is clear that the operators Ω_j leave the spaces $\mathcal{G} = \ker b = \ker c$ and $\mathcal{H}$ invariant.

We shall show that all $\Omega_j = 0$ on $\mathcal{H}$ for $j = 1, \ldots, 4$. First note that $\Omega_4 : \mathcal{H}_n \to \mathcal{H}_n$, because of the relation $\Omega_4 bc = bc\Omega_4$ and the fact that $\mathcal{H}_n = \ker(bc + q^{2n+1} I)$. Since $\Omega_4 b = \epsilon q b \Omega_4$, we therefore obtain that $w(\Omega_4 \lceil \mathcal{H}_n) w^* = \epsilon q(\Omega_4 \lceil \mathcal{H}_n)$, so that $\Omega_4 \lceil \mathcal{H}_n = 0$ by Lemma 6 and hence $\Omega_4 = 0$ on $\mathcal{H}$. Consequently we have $\Omega_3 bc = bc\Omega_3$ which implies that $\Omega_3 : \mathcal{H}_n \to \mathcal{H}_n$. Since $\Omega_4 = 0$, we have the two relations

$$\Omega_1 a = \epsilon q a \Omega_1 + \epsilon b \Omega_3, \tag{109}$$

$$\Omega_1 b = \epsilon q^{-1} b \Omega_1 + \epsilon a \Omega_2. \tag{110}$$

Using the fact that $\Omega_3 : \mathcal{H}^n \to \mathcal{H}^n$ it follows from (110) by induction that $\Omega_1 : \mathcal{H}^n \to \mathcal{H}^{n+1}$. Because $\Omega_1^* = -\Omega_1$, we have $\Omega_1 : \mathcal{H}_n \to \mathcal{H}_{n-1} \oplus \mathcal{H}_n \oplus \mathcal{H}_{n+1}$. Hence Ω_1 is of the form

$$\Omega_1 \eta_n = A_n \eta_{n-1} + B_n \eta_n + A_{n+1}^* \eta_{n+1}, \qquad \eta \in \mathcal{H}_0,$$

where A_n and B_n are bounded linear operators on $\mathcal{H}_0$. Inserting this expression into (110) and comparing the nth components, we get $q B_n w^* = \epsilon w^* B$. Thus, $w B_n w^* = \epsilon q^{-1} B_n$ and hence $B_n = 0$ by Lemma 6. Comparing the $(n-2)$th components in (109), we obtain the equation $\lambda_n A_{n-1} = \epsilon q \lambda_{n-1} A_n$, so that we have

$$A_n = (\epsilon q)^{1-n} \lambda_n \lambda_1^{-1} A_1. \tag{111}$$

Since $\|A_n\| \le \|\Omega_1\|$ and $q^{-n}\lambda_n \to +\infty$ if $n \to \infty$ (recall that $0 < q < 1$), we conclude from (111) that $A_n = 0$ for all $n \in \mathbb{N}$. Thus $\Omega_1 = 0$ on $\mathcal{H}$. Applying once more Eqs. (109) and (110), we see that $\Omega_2 = \Omega_3 = 0$ on $\mathcal{H}$.

A simpler reasoning shows that the operators Ω_j, are also zero on $\mathcal{G}$. Since the four 1-forms ω_j, $j = 1, \ldots, 4$, generate the $4D_\pm$-calculus as a left $\mathcal{A}$-module, it follows that $\mathrm{d}_{\pi, F}(x) = 0$ for all $x \in \mathcal{A}$.

The case of the $3D$-calculus is even much simpler. Recall that $\omega_j b = \alpha_j b \omega_j$ and $\alpha_j \omega_j a = a\omega_j$ by (6) and (7), where $\alpha_0 = \alpha_2 := q$ and $\alpha_1 := q^2$. These relations imply that $w\Omega_j w^* = \alpha_j \Omega_j$ on $\mathcal{H}$ and $\alpha_j \Omega_j = v\Omega_j v^*$ on $\mathcal{G}$. Therefore, by Lemma 6 we get $\Omega_j = 0$ on $\mathcal{H}$ and on $\mathcal{G}$ for $j = 0, 1, 2$. This completes the proof of Theorem 3.

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